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Large fluctuation of thermodynamic quantities in a coupled Bernoulli map lattice

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The two-dimensional Ising model is studied with a coupled Bernoulli map lattice model. The large fluctuation of some thermodynamic quantities such as energy, the Kolmogorov-Sinai entropy, and magnetization is calculated with the characteristic function. The average value of energy and magnetization at different coupling constants can be estimated from the large fluctuation of the thermodynamic quantities at a fixed coupling constant. [S1063-651X(98)08307-X]

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Coupled map lattice models have been used to study chaotic spatio-temporal patterns [1–5]. We proposed a coupled map lattice that exhibits a statistical-mechanical phase transition [6,7]. Using the coupled map lattice, we can perform a simulation for the phase transition of the two-dimensional Ising model, in which several thermodynamic quantities are exactly expressed [8].

On the other hand, the large deviation theory [9] has been applied to the multifractal [10], statistical properties of turbulence [11] or chaos [12,13]. We also calculated the large fluctuation of the Ising model with mean-field coupling using the Renyi entropy. Then, we discussed a relation between the large fluctuation of some thermodynamic quantities at a fixed temperature and average values of the thermodynamic quantities at different temperature [14].

In this Brief Report, we calculate the large fluctuation of some thermodynamic quantities for the two-dimensional Ising model with the coupled Bernoulli map lattice.

Each elemental map is the Bernoulli shift,

$$X_{n+1}=\frac{2}{1+a}(X_n+1)-1 \quad \text{for} \quad -1<X_n<a,$$
$$X_{n+1}=\frac{2}{1-a}(X_n-1)+1 \quad \text{for} \quad a<X_n<1, \tag{1}$$

where a is a parameter satisfying $-1<a<1$. The Bernoulli shift has a uniform invariant measure $\rho(X)=1/2$ over $-1<X<1$. A spin variable s_n is defined as $s_n=\mathrm{sgn}(X_{n+1}-X_n)$; that is, $s_n=1$ for $X_n<a$ and $s_n=-1$ for $X_n>a$. The mean value of s_n is a and its time correlation is $\langle s_n s_m\rangle=\delta_{n,m}$. That is, the Bernoulli shift (1) can work as a random number generator.

We construct a two-dimensional coupled map lattice composed of the Bernoulli shift,

$$X_{n+1}^{i,j}=\frac{2}{1+a_n^{i,j}}(X_n^{i,j}+1)-1 \quad \text{for} \quad -1<X_n^{i,j}<a_n^{i,j},$$
$$X_{n+1}^{i,j}=\frac{2}{1-a_n^{i,j}}(X_n^{i,j}-1)+1 \quad \text{for} \quad a_n^{i,j}<X_n^{i,j}<1, \tag{2}$$

where $1\leqslant i\leqslant L$ and $1\leqslant j\leqslant L$ denote the lattice points in the $L\times L$ square lattice. The parameter $a_n^{i,j}$ is a time dependent variable expressed as

$$a_n^{i,j}=\tanh\left\{\frac{K}{4}(s_{n-1}^{i+1,j}+s_{n-1}^{i-1,j}+s_{n-1}^{i,j+1}+s_{n-1}^{i,j-1})\right\}, \tag{3}$$

where K is a coupling constant.

As shown in a previous paper, the probability $p_n(\{m^{i,j}\})$ that the spin configuration $\{s_n^{i,j}\}$ takes $\{m^{i,j}\}$ at time n obeys a master equation,

$$p_n(\{m^{i,j}\})=\sum_{m'^{i,j}} p_{n-1}(\{m'^{i,j}\})w(\{m'^{i,j}\}\rightarrow\{m^{i,j}\}). \tag{4}$$

The transition probability $w(\{m'^{i,j}\}\rightarrow\{m^{i,j}\})$ is expressed as

$$w=\prod_{i,j}\tfrac{1}{2}[1+\tanh\{(K/4)m^{i,j}\times(m'^{i+1,j}+m'^{i-1,j}+m'^{i,j+1}+m'^{i,j-1})\}]. \tag{5}$$

As in a previous paper, we consider the checkerboard type updating rule for the coupled map lattice; that is, the updating is performed alternatively for even lattice points where $i+j$ is even and odd lattice points. In this case a detailed

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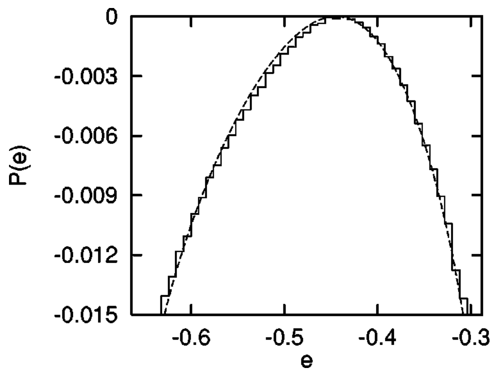

FIG. 1. The generalized entropy $P(e)$ for the energy. The solid line is obtained by the characteristic function and the dotted line is the logarithm of the histogram of e.

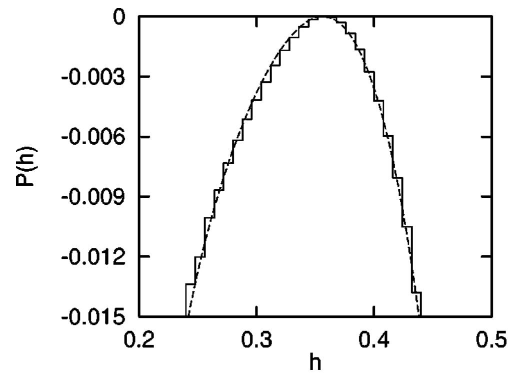

FIG. 2. The generalized entropy $P(h)$ for the KS entropy. The solid line is obtained by the characteristic function and the dotted line is the logarithm of the histogram of h.

balance condition is satisfied for the Markov process (5) and the equilibrium distribution p_{eq} is obtained as

$$p_{eq}(\{m^{i,j}\}) \propto \exp\left\{ (K/8)\sum_{i,j} m^{i,j} \times (m^{i+1,j}+m^{i-1,j}+m^{i,j+1}+m^{i,j-1}) \right\}, \quad (6)$$

which is equivalent to the equilibrium distribution of the two-dimensional Ising model. The Ising system exhibits a phase transition at $K=2\ln(1+\sqrt{2})$ for $L\rightarrow\infty$.

The thermodynamic quantities such as the energy $E=-(K/8)\Sigma_{i,j}s^{i,j}(s^{i+1,j}+s^{i-1,j}+s^{i,j+1}+s^{i,j-1})$ and the magnetization $M=\Sigma_{i,j}s^{i,j}$ are quantities of $O(N)$ where $N=L\times L$ and the probabity $p(Q)$ that a thermodynamic quantity takes Q which is deviated from the average value behaves as $p(Q)\sim\exp[NP(q)]$, where $P(q)$ is an exponent that denotes the decrease rate of the probability of the large deviation as the system size N and q is the thermodynamic quantity per spin $q=Q/N$. The quantity $-P(q)$ corresponds to the free energy in the statistical mechanics and it is called a rate function in the large deviation theory. We call $P(q)$ a generalized entropy in this paper. The generalized entropy can be calculated from a characteristic function $\phi(\mu)=\langle\exp(\mu Q)\rangle$, where $\langle\cdots\rangle$ denotes the average with respect to the equilibrium distribution. If N is large enough, the characteristic function can be estimated at $\phi(\mu)\sim\exp\{N[\mu q_\mu+P(q_\mu)]\}$ by the saddle point method, where q_μ satisfies $\partial P(q_\mu)/\partial q=-\mu$. If we define $\psi(\mu)=(1/N)\ln\phi(\mu)\sim\mu q_\mu+P(q_\mu)$, the generalized entropy function and ψ obey

$$\frac{\partial\psi}{\partial\mu}=q_\mu, \quad P(q_\mu)=\psi(\mu)-\mu q_\mu. \quad (7)$$

If we obtain $\psi(\mu)$ from a numerical simulation, we can construct the generalized entropy $P(q)$ from Eq. (7). We have performed a numerical simulation of the coupled Bernoulli map lattice with $L=25$ and $K=1.6$. As an initial condition, $x^{i,j}$ are randomly distributed between -1 and 1 and $s^{i,j}$ is assumed to be 1. The total time step is 2×10^6 and we have calculated time average instead of the thermal average $\langle\cdots\rangle$, and further we have taken sample average of ten samples where the initial conditions of $x^{i,j}$ are different. Figure 1 displays the generalized entropy for the energy $e=E/N$ per site, which is calculated from the characteristic function and the logarithm of the histogram of the probability distribution for E/N which is directly obtained from the simulation. As N is larger, the estimate of the characteristic function at the maximum point becomes better, however, the occurrence of the large deviation of the thermodynamic quantity becomes very rare. So the simulation at moderate size N is suitable.

Our dynamical system is chaotic, and a quantity which characterizes the dynamical chaos is the KS entropy. In our model the KS entropy is represented as

$$H_{KS}=\sum_{i,j}\left\langle -\tfrac{1}{2}(1+a_n^{i,j})\ln\{\tfrac{1}{2}(1+a_n^{i,j})\} -\tfrac{1}{2}(1-a_n^{i,j})\ln\{\tfrac{1}{2}(1-a_n^{i,j})\}\right\rangle, \quad (8)$$

which is the thermal average of the sum of the local Liapunov exponent at each site. This quantity is equivalent to

$$H_{KS}=-\sum_{m'^{i,j}}\sum_{m'^{i,j}} p_{eq}(\{m'^{i,j}\})w(\{m'^{i,j}\}\rightarrow\{m^{i,j}\}\ln[w(\{m'^{i,j}\}\rightarrow\{m^{i,j}\}]. \quad (9)$$

The KS entropy is a thermodynamic quantity and it seems to exhibit a singularity at the phase transition point as shown in [15]. Figure 2 displays the generalized entropy function for the KS entropy per site and the corresponding logarithmic histogram of $h=H_{KS}/N$. Good agreement is seen in Figs. 1 and 2 for $L=25$.

We can also calculate more generalized entropy function $P(q,r)$ for the combination of the two thermodynamic quantities $Q=Nq$ and $R=Nr$ from the corresponding characteristic function $\phi(\mu,\nu)=\langle\exp\{N(\mu q+\mu r)\}\rangle$. In particular, we consider the case $r=e=E/N$. In the thermal equilibrium state, the probability distribution function is written as $p_{eq}\propto\exp[N\{-e/T+S(q,e)\}]$, where T is the temperature and $S(q,e)$ is the thermodynamic entropy function. The characteristic function $\phi(\mu,\nu)=\langle\exp\{N(\mu q+\nu e)\}\rangle$ can be estimated by the saddle point method as

$$\exp[N\{\mu q_{\mu,\nu}+\nu e_{\mu,\nu}-e_{\mu,\nu}/T+S(q_{\mu,\nu},e_{\mu,\nu})\}], \quad (10)$$

where $q_{\mu,\nu}$ and $e_{\mu,\nu}$ satisfy

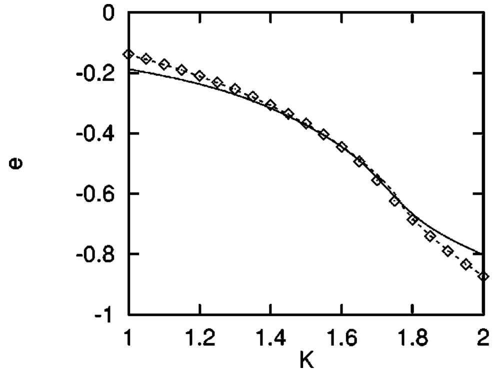

FIG. 3. Energy e as a function of K. The solid line is obtained from the characteristic function at a fixed $K=1.6$, the points denote numerically obtained values by changing K, and the dotted line is the exact solution of the energy for the infinite sytem.

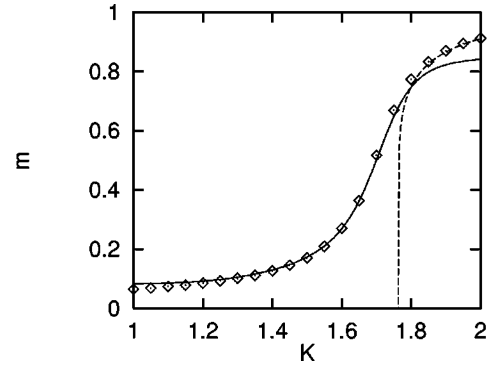

FIG. 4. Magnetization m as a function of K. The solid line is obtained from the characteristic function at a fixed $K=1.6$, the points denote numerically obtained values by changing K, and the dotted line is the exact solution of the magnetization for the infinite sytem.

$$\frac{\partial S}{\partial q}(q_{\mu,\nu},e_{\mu,\nu})=-\mu,\quad \frac{\partial S}{\partial e}(q_{\mu,\nu},e_{\mu,\nu})=\frac{1}{T}-\nu. \tag{11}$$

The logarithm of the characteristic function $\psi(\mu,\nu)=1/N\ln\phi(\mu,\nu)$ and the entropy function satisfy

$$\psi(\mu,\nu)=\mu q_{\mu,\nu}+(\nu-1/T)e_{\mu,\nu}+S(q_{\mu,\nu},e_{\mu,\nu}),$$

$$\frac{\partial\psi}{\partial\mu}=q_{\mu,\nu},\qquad \frac{\partial\psi}{\partial\nu}=e_{\mu,\nu}, \tag{12}$$

$$S(q_{\mu,\nu},e_{\mu,\mu})=\psi(\mu,\nu)-\mu q_{\mu,\nu}-(\nu-1/T)e_{\mu,\nu}.$$

The equation $\partial S/\partial e=1/T-\nu$ in Eq. (11) implies that the entropy function gives information at different temperature T', where $1/T'=1/T-\nu$. That is, $q_{\mu,\nu}$ at $\mu=0,\nu=1/T-1/T'$ is equal to the average value of the thermodynamic quantity Q/N per site at temperature T'. The quantity $q(0,\nu)$ is given by $\partial\psi/\partial\mu$ at $\mu=0$ and ν .

We have reconstructed two thermodynamic quantities: the magnetization and the energy at different coupling constants from the characteristic function $\phi(\mu,\nu)$ at a fixed value of $K=1.6$. Figure 3 displays the reconstructed value of the energy $-(K/8N)\Sigma_{i,j}s^{i,j}(s^{i+1,j}+s^{i-1,j}+s^{i,j+1}+s^{i,j-1})$, which is calculated as

$$e(K')=\frac{K'}{KN}\frac{\partial}{\partial\nu}\ln\langle\exp(N\nu e)\rangle_{\nu=1-K'/K}.$$

The points show the energy for several values of K, which is obtained by directly changing K and taking the average of E for the coupled Bernoulli map lattice of $L=25$. The dotted line denotes the exact solution for the infinite system; that is,

$$e=-(K/4)\cosh(K/2)/\sinh(K/2)(1+2/\pi\kappa''K_1), \tag{13}$$

where $\kappa''=2\tanh^2(K/2)-1$ and K_1 is the complete elliptic integral at $\kappa=2\sinh(K/2)/\cosh^2(K/2)$. The reconstructed curve is close to the directly obtained values and the exact solution for $1.4<K<1.8$. Figure 4 displays the reconstructed curve of the magnetiztion $m=(1/N)|\Sigma_{i,j}s^{i,j}|$, which is calculated as $m(K')=(1/N)(\partial/\partial\mu)\ln\langle\exp\{N(\mu m+\nu e)\}\rangle_{\mu=0,\nu=1-K'/K}$, the directly obtained values, and the exact solution for the infinite-size system:

$$m=0\quad\text{for}\quad K<K_c,$$
$$=\left\{1-\frac{1}{\sinh^4(K/4)}\right\}^{1/8}\quad\text{for}\quad K>K_c. \tag{14}$$

The directly obtained values are deviated from the exact solution owing to the finite-size effect; however, the reconstructed curve fits the directly obtained values fairly well for $K<1.9$. We cannot reconstruct the KS entropy at different coupling constants from the data at a fixed coupling constant, because the coupling constant K is involved in the $a_{i,j}$ in Eq. (3) in a complicated manner.

To summarize, we have applied the thermodynamic formalism to the two-dimensional coupled Bernoulli map lattice and characterize the large fluctuation of some thermodynamic quantities. The coupled Bernoulli map lattice is a toy model, however, an instructive model that connects the chaotic dynamics and the statistical mechanics.

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